ALGEBRAIC MANIPULATION: ACTIONS, RULES AND RATIONALES.

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Twenty year-10 students in a boys' private school in Melbourne were interviewed individually while they worked on algebraic tasks. The tasks included solving very simple linear equations and simplifying expressions. The taped interviews provide examples of informal, and often misleading, ways in which many students describe algebraic manipulation procedures. Informal terminology (e.g., "You move the 2 over there and put it on top") was used successfully by some students, but for many others it was associated with making mistakes. Students often appeared to be guided by memory of actions they might carry out rather than by general principles. They showed difficulties understanding the ways in which numbers can be used for checking algebraic work and the purposes of basic algebraic tasks such as simplifying and solving.

In no part of mathematics is the tension between rule and understanding stronger than in algebra. Long-standing research shows that error-free manipulation is rare, errors such as $3^2 = 6$, $4(n+5) = 4n+5$, $(x+8)/(x+2) = 8/2$ being common (Carry, Lewis and Bernard, 1980; Kuchemann 1981, Foxman et al, undated). Other work has shown that even 16 year-old students whose manipulative skills are good have very little understanding of how to use algebra in the processes of generalising, reasoning or justifying statements (Lee & Wheeler, 1987). Lee and Wheeler (1987) have also pointed out the fragility of the connection that many students make between arithmetic and algebra. There are also indications of deep-seated misinterpretations of the meanings of algebraic symbols. For example, many students regard an equation such as 6s=p as a description of a loose association between quantities (e.g. a comparison of relative sizes) rather than as a strict equality between numbers. (Clement, 1982; MacGregor, 1991). The tensions between rule and understanding are evident in all these instances.

The focus of the present work is on performance in algebraic manipulation involving several steps. The context in which the work is carried out is that of comparison between expert and novice performance in algebraic manipulation, in particular in solving linear equations and simplifying expressions. Our research aims to understand how the additional conceptual awareness of experts develops and how it relates to performance skills based on symbol movements. We seek to understand how students' understanding commonly breaks down in the course of this development and how such breakdowns might be minimised. In the present paper, we present extracts from interviews with Year 10 students which illustrate the following three aspects of this problem.

(i) Students seem to be guided by images of actions which are not related to underlying general principles. This is reflected in the language of actions and goals used by both students and teachers to discuss algebraic transformations. Some students denied that they work with general principles, but rather claimed that they could just see what to do.

(ii) Students showed a lack of understanding of the rationale of using numbers to check algebraic results. The extracts from the interviews illustrate three different modes of checking which use numbers in different ways and there are probably others.

(iii) The purpose of algebraic tasks is unclear and students lose sight of the aims as they work. Even within the purely symbolic tasks that are reported in this paper, one student wavered between simplifying and solving, using actions appropriate for one in the other.

The data used here was collected as part of an experiment focused on algebraic manipulation procedures used for solving one- and many-variable equations, and, to a smaller extent, on the forming of equations. Two classes of

Year 10 boys ('middle' sets in a relatively 'good' school) were taught these topics for three weeks by the first author, using a method in which the checking of procedures was an integral part of the work. Errors were immediately rectified or discussed and students generated and checked their own examples. Concurrently, some twenty of the students were being interviewed on video by the first author, one to one, on the material being taught, to expose more fully their methods and the justifications which they might have for them.

In this paper we report extracts from interviews with two of the students, which illustrate behaviour that was commonly observed. About one half of the interview with Andrew is in Appendix 1, a further extract is in Appendix 2 and an extract from Steven's interview is Appendix 3. In the extract in Appendix 1, Andrew makes a manipulation error which was observed in several interviews. He knows that to "undo" the division by 2 on the left hand side, he must multiply. However, he *replaces* the division by multiplication, thus in effect multiplying the first term by 4. An attempt to clarify the error by looking at numerical fractions is made difficult by the fragility of Andrew's knowledge of fractions. Andrew checks a solution to an equation by substituting the number found throughout the whole equation and then repeating the steps of the algebraic solution with the numbers. In Appendix 2, we see Andrew using false cancelling to simplify an expression and, as he tries to check it numerically, possibly confusing simplifying with solving. In Appendix 3, Steven shows a similar confusion and also illustrates strongly how his work is guided by images of moving numbers and symbols rather than by principles.

ACTIONS, GOALS AND PRINCIPLES.

It is clear that perceptions of symbol movements play a part in algebraic manipulation carried out by both experts and novices. For example, in a recent study (Bell & Malone, 1992) of approaches to the reading and writing of algebraic formulae, students were asked to construct and select valid rearrangements of the formulae C = V ÷ R (for current, voltage and resistance), S = D ÷ T (speed, distance and time) and A = B + C (three numbers). Later in the test, they were asked to say which most closely described their way of thinking about these questions:

> Look back at the question on C, V and R. Tick the one or two of the following statements which is closest to the way you were thinking.
> a. I thought which would be the biggest number, so the others would be divided into it.
> b. I tried some actual numbers in my head.
> c. I just remembered the formulae.
> d. I thought the one on top of the right hand side would go underneath on the other side.

Of the school pupils, aged 13-15, about 25% reported thinking which was the bigger number, 40% reported trying actual numbers, 20% just remembered the formulae, and only 10% reported thinking of physical movement of the symbols. But of the two samples of experts (teachers or prospective teachers in training) the great majority reported thinking of physical movements, and none considered numbers or sizes. (Bell & Malone, 1992).

In contrast to the Bell and Malone sample of students, the students we interviewed in this study seemed to think predominantly in terms of symbol movements. In Appendix 1 (interchange 2), Andrew makes a manipulation error which was observed in several interviews. To "undo" the division by 2 on the left hand side, he *replaces* the division by multiplication, thus in effect multiplying the (f - 3) by 4. When the interviewer tries to clarify what has gone wrong (interchanges 10 - 31), note the confusion that is shown by Andrew between multiplying and dividing, especially in the denominator (interchanges 11 - 14) and the interviewer's "action-oriented" language (multiply, divide, throw away: interchange 15). In discussing algebraic manipulation, most of our language is like this, stressing goals and actions, rather than the principles involved. We do not seem to have alternatives readily available. We had hypothesised that differences would be observable between the students' and the interviewer/teacher's language, in that students' language would be purely descriptive of the surface aspects of the procedures (e.g. "Move the 3 over here") while the teacher's language would refer to underlying principles and justification (e.g. ""We must do the same thing to both sides, to keep the balance"). Study of the complete interview records shows that both teacher and students spoke primarily in terms of surface aspects of symbolic procedures, although the teacher's talk contained the additional dimension of validity or justification, drawing a distinction between the goal (e.g., "I want to get rid of the 3 in the denominator") and the method of achieving that goal (e.g., "I will multiply both sides by 3"). The students' talk often referred only to the goal, and when a general rule was quoted, which was not often, it would be expressed in terms of "what you have to do" rather than what is a valid operation on the equation or fraction. Both teacher and students speak about symbol movements such as "getting rid of the denominator", but for the teacher there is a distinction between goals and valid methods for moving towards these goals whereas for the students there appears to be no such distinction.

All of the errors seen in here in Andrew's work were common to many of the subjects. We sought for clues as to their origin in the language used. Note, for example, that when Andrew was asked if he was using a rule, he does not speak of doing something to both sides but says he was "times-ing everything along there by 2" and says he is not sure whether he should multiply the *f* term (interchanges 7 - 10). He appears to have a visual image of what he should do rather than an explicit rule to apply. Another boy said of the 2 in the denominator, "It's divided, so you have to multiply". Other boys' words and handwaves seemed to suggest that it was that particular 2 on the bottom which was being used to multiply the top (or to multiply the number on the right hand side), so it was thus felt to have been appropriately used up and did not still remain as a denominator.

The interviewer frequently asked the boys about the rules they used. Most of them denied having any rules governing their manipulations, as for example does Steven in Appendix 3, interchanges 2 - 4. Another boy, in a similar situation, said " No, I just do it. Rules just go right over the top [of my head]." One of the few boys who did talk about doing things to both sides of an equation actually failed (or forgot) to do so. He professed to subtracting 4 on each side, then "times-ing" both sides by 3, and in both cases actually did it only on the left hand side until reminded by the interviewer. He also collected terms onto one side by simply moving them, that is, by bringing a term to the other side of the equation without changing its sign. It is clear that the ability to state a correct rule does not guarantee its correct application. Rules need to be taught along with discussion of the ways in which they may be applied, with exposure to possible errors and, of course, with practice.

USING NUMBERS TO CHECK ALGEBRAIC PROCEDURES.

During the twenty interviews, many students showed they were confused about the different ways in which numbers are used to check the results and validity of algebraic procedures. In fact, there are several conceptually different ways in which this must be done and three can be seen in the extracts from interviews in the Appendices. In the first segment of the interview with Andrew in Appendix 1 (interchange 4) we see an instance where a solution of an equation is being checked. The checking procedure that Andrew uses, which is to repeat the complete solving process with numbers, fails because Andrew repeats his algebraic error with the numbers. A similar checking procedure was also observed in several other interviews. Later on (Appendix 1, interchange 15 *et seq*), we see a quite different use of numbers for checking algebra, when the interviewer tries to illustrate, almost by analogy, that when $(f - 3)/2$ is multiplied by 2, the answer is not $2(f - 3)$. Unlike the equation solving where only the solution is to be used in the check, the interviewer chooses an arbitrary fraction, in this case 3/4. Andrew's vagueness may indicate that he does not comprehend the relevance of the interviewer's questions about 3/4 to his problem about $(f - 3)/2$. The reference to a numerical fraction was an attempt by the interviewer to find some basis for demonstrating to Andrew that this was an incorrect operation and to enable him to replace it by the correct one. It ran into trouble because his conception of fraction multiplication by an integer was faulty - he thought you multiply both the top and the bottom. This error is also shared by many of the Year 10 students interviewed. However it is worth noting that this does not correspond to Andrew's algebraic misconception so, on this evidence, we see that he does not perceive a link between the arithmetic and the algebra. The attempt to link with the real-life situation of four times 3/4 of a cake does not go easily either; it does not seem a natural recourse for him.

Andrew's interview continued (Appendix 2) with questions about simplifying expressions. He appeared at first (interchange 2) not to be entirely clear about what this meant, and how it was different from the previous kinds of manipulation (i.e., solving equations). Note how the checking that he is doing here uses numbers in yet another way. Andrew does not appreciate that any number will do for checking whether something is an identity (in this case whether $(2x+6)/3$ is equal to $2x+2$). In fact, to be precise in this case, any number *other than* the solution of the equation $(2x+6)/3 = 2x+2$ will do, but by the end of the episode, Andrew seems to be searching for this solution.

After the twenty interviews with boys from the middle Year 10 sets, six further interviews were conducted with boys from the top set of the same year. Some of them used the same language as the boys in the first sample, and made the same mistakes though perhaps not so frequently. The most noticeable difference was the much more ready way in which they recognised and were able to pick up and use the checking and verifying procedures suggested by the interviewer.

SOLVING OR SIMPLIFYING.

Andrew made a false cancellation of the 6 and the 3 (Appendix 2, interchange 4) He was then guided by the interviewer through a checking procedure, substituting a number for x. He appeared not to understand the rationale for this initially, but did agree that the numerical result from the two expressions should be the same. But when they were not, his suggestion for remedying the situation was to try a different value for x, only to find that it still doesn't work out. (interchanges 12 - 13) Note the language here; he appears now to think that the task is to solve an equation, i.e. find a value for x which will equate the two expressions. Simplifying has mutated to solving, during the checking episode. Steven also shows (Appendix 3, interchanges 6 - 10) how symbol moves which are appropriate to simplifying tend to be employed in a solving situation, although there are other explanations for these moves.

CONCLUSIONS

To summarise, we suggest that major causes of errors in manipulation in algebra for these students include the rules about how to "move" numerals and letters, that they have developed by generalising from simple cases in early algebra learning and now remember as actions to be carried out, rather than as principles. The memory of spatial patterns associated with the fraction bar and the equals sign, which guide experts, can easily be misused by novices. Even when verbal rules are remembered by rote they may be given no meaning, and therefore be unusable.

The implications for teaching are, we suggest, that there needs to be much more exploratory work, in which possible rules and interpretations are experimented with and their validity checked by such methods as substituting numbers. Explicit attention needs to be directed to the different underlying logic that is involved when numbers are used to check algebra. There needs also to be, of course, practice to gain fluency - but checking in its many forms should be built in as a natural part of the task.

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Appendix 1: Interview with Andrew

I.(1) Can you solve this equation?

A.(2) *Andrew, working silently, multiplies all the terms of the equation, but when multiplying the fraction, multiplies the top as well as removing the 2 in the denominator.*

$$\frac{f-3}{2} + 4f = 2$$
$$2f - 6 + 8f = 4$$
$$10f - 6 = 4$$
$$10f = 10$$
$$f = 1$$

I.(3) Right, will you now check your answer?

A.(4) *When he comes to check the solution, A. writes the whole equation, including the right hand side, and repeats his solution procedure. Thus his check repeats his error.*

$$\text{LHS} = \frac{1-3}{2} + 4 = 2$$
$$= 2 - 6 + \cancel{1}8 = 2$$
$$4 = 2$$

I.(5) Oh, so you've got 4=2. If you are checking, do the left hand side on its own like this (Interviewer indicates one side of the equation and writes *LHS = on the left.*

A.(6) *A crosses out =2 and starts a fresh check, working out the LHS only, but still performs the same operations.*

$$\frac{1-3}{2} + 4$$
$$2 - \cancel{6} + 8$$
$$= 4$$

This doesn't equal the right hand side so I. asks

I.(7) Have you any idea what's wrong? *(Little response)* Look here, (*pointing to the top line*) you had f - 3 over 2 and in the next line you had 2f - 6. What equation rule were you using at this point?

A.(8) I was times-ing everything along there by 2 (*waving his hand over the whole equation).*

I.(9) You multiplied everything there by 2....?

A.(10) I'm not sure whether I should have multiplied the f or not by 2.

I.(11) Well you've multiplied the 4f by 2 and the right hand side by 2 and that's all correct. But multiplying this fraction by 2 you've done wrongly. You multiplied the top by 2; and what have you done with the bottom?

A.(12) Multiplied that by 2

I.(13) But that would be 4, wouldn't it?

A.(14) But isn't that divided?

I.(15) *(Pointing to the very first line)*What this says is, f - 3 divided by 2. You've multiplied it by 2 *and* thrown away the bottom. That's equivalent to multiplying the top by 2 and dividing the bottom by 2.
If that was a fraction like 3/4 ... . If I multiply 3/4 by 4, what do I get?

A.(16) 12/16

$$\left(\frac{3}{4} \times 4\right) \quad \frac{12}{16}$$

I.(17) What's the value of 12/16?

A.(18) 3/4

I.(19) Well if that's *equal* to 3/4, it can't be *multiplied by* 4, can it?
Think of it in real life, ...in ordinary language, what is 3/4 times 4?

A.(20) *No response.*

I.(21) 3/4 of a cake, multiply it by 4, how much cake would you have?

A.(22) You'd get four 3/4s.

I.(23) How much would that be altogether?

A.(24) It would be 3. 3 cakes.

I.(25) How many quarters altogether?

A.(26) 16.

I.(27) How many quarters in 3?

A.(28) 12.

I.(29) So when you do this(*3/4* x *4)* this had better be 12 quarters, or 3. *(I writes this)*
So what does this have to come to? What rule should you use when you are working this out?

A.(30) Multiply the top. *(Looking at the writing from interchange 16 as above and comparing with the 12 just written in interchange 29.)*

I.(31) Yes, so when you multiply this *((f-3)/2)* by 2 you multiply the top by 2 and leave the bottom. That's what went wrong here (*pointing to the top line again*). So that sorts out an important problem for you.

Appendix 2: Andrew's interview continues.

I.(1) Now can you simplify that? i.e. (2x+6)/3

A.(2) What? Do you mean like I did the last one?

I.(3) No, replace it by a simpler expression.,..like if I said 4/2, and asked you to simplify that.

A.(4) *A. cancels 3s from the 3 and the 6 and gets 2x + 2 over 1, then 2x +2*

$$\frac{2x+\cancel{6}}{\cancel{3}\,1} \quad \frac{2x+2}{1}$$

I.(5) Could you put a value in for x in this, and in this, to check?

A.(6) *A puts in 3 and does as before, getting 8.*

$$\begin{array}{c} 2x+2 \\ 6+2=8 \end{array}$$

I.(7) Is there any other way you could work this out?

A.(8) Do you mean with an x or something?

He works out (2x+6)/3 getting 12 / 3 and then 4.

$$\frac{12}{3} \; 4 \; \frac{2x+6}{3}$$

I.(9) Should this be the same?

A.(10) Yes.

I.(11) What's wrong?

A.(12) You could try putting a smaller number, to get it smaller - put 1 instead of x.

He does this and of course the expressions still do not tally.

$$\frac{2+6}{3} \; \frac{6}{3} \; 2+2=4$$

I.(13) So it still doesn't work out.

Appendix 3: Extract from Steven's interview.

I.(1) *Presents Steven with this equations and asks* Any ideas?

$$\frac{a-3}{4} = \frac{a+2}{3}$$

Steven. (2) Yes. Put that in brackets and multiply that by 3, and multiply that by 4.
He points to the left hand side and the right hand side in turn.

I.(3) Do you have any rule that you are using for that? Like cross multiplication or something?

S.(4) No, I just know that - you know..(*Pointing with his finger to the two sides.)*

I.(5) *Interviewer I gives him a new equation.* Would you do the same thing here?

$$\frac{a-3}{4} = \frac{a+2}{3} + 1$$

S.(6) I think I would.

$$3(a-3) = 4(a+2)+1$$
$$3a-9 = 4a+8+1$$

S.(7) Would that equal zero, -9 +9?

I.(8) How do you mean?

S.(9) I think that -9 +9, oh! I think that's where I made my mistake.

I.(10) Yes, you've added the things on the two sides.
You have to think of this as two equal things on the two sides.
Pointing to the two sides of the equation.

S.(11) I was thinking you add them together.

I.(12) Yes, but if you add 9 to this side, you've got to add 9 to that side as well.

S.(13) Yes. I think I know where I've gone wrong.

I.(14) Yes, we can sort that out. There's something gone wrong up here too.
I. points to line X.
When you do this multiplication across like this it's only going to be valid if you can justify it by saying you have multiplied both sides by the same thing, and I don't think you have.

S.(15) Oh - I understand. So I have to multiply this by 3 as well. *Pointing to 4(a + 2).*

I.(16) No, what we have to do is multiply both sides by 12 and we've got to multiply this (*pointing to the 1 on the right hand side)* by 12 as well.

S.(17) So this has got to be 12.
He turns the 1 into a 12 and continues

$$3(a-3) = 4(a+2) + 12$$
$$3a - \cancel{9} = 4a + 8 + 12 + 9$$
$$\cancel{7a}\ 3a = 4a + 29$$

S. (18) So I've just got to add 9 to this and to that...now I minus this one *(pointing to the 4a)*

I.(19) Yes.

With some help, Steven works this out and reaches -a = 29:, answer a = -29: then, with more help, makes a correct check.
